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Weakly nonlinear stability of Hartmann boundary layers

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Abstract

By means of a weakly nonlinear stability analysis it is shown that the Hartmann boundary layer presents subcritical instability in the proximity of the minimum linear critical Reynolds number. This gives further support to earlier speculations that finite amplitude effects account for the discrepancies between the results of the linear stability analysis and experimental evidence on laminarisation.

Keywords: Transition; Hartmann layers; Weakly nonlinear; Magnetohydrodynamics

1. Introduction

Magnetic fields are used in many engineering applications that involve electrically conducting fluids. They are employed, for example, to drive flows, induce stirring, levitation or to suppress turbulence. Applications are found in the casting of metals and the growth of semiconductor crystals [1]. In many of those magnetohydrodynamic flows the balance of Lorentz and viscous forces along boundaries non-tangential to the magnetic field gives rise to Hartmann boundary layers. These layers act as a channel for the electric currents and determine the velocity in the core of the flow in the laminar regime [2].

A linear stability analysis of the Hartmann layer was first carried out by Lock [3] neglecting the Lorentz force term in the disturbance equations. The complete equations were later solved by Lingwood and Alboussière [4] who found a critical Reynolds number (based on the Hartmann layer thickness) of $R_c \approx 48250$. On the other hand, the laminarisation of magnetically driven flows in ducts of rectangular cross section is experimentally observed to occur in the range $150 < R_c < 250$ [5]. If transition in the Hartmann layers is to account for these observations, a plausible explanation for the discrepancy with the predictions of the linear theory is that it is the result of nonlinear effects. While for these values of the Reynolds number the linear analysis predicts stability to infinitesimal disturbances, the flow may be unstable to small but finite size disturbances (subcritical instability). In this case, there will be a lower bound in the amplitude of the disturbance for the instability to develop. Recent studies [6] have shown that even if the norm of the perturbations is below this threshold value, if the linear stability operator is non-normal, they may still be amplified in the subcritical region of parameter space. Although this growth is only transient, it may be strong enough for the amplitude of the perturbations to reach this critical value and trigger nonlinear effects.

Information on the first stages of the evolution of the perturbed flow can be obtained with a weakly nonlinear stability analysis. As described by Stuart [7], an amplifying disturbance in a laminar flow may reach such a magnitude as to distort the basic flow and hence alter the exchange of energy between it and the perturbation. By energy arguments he showed that, near the linear critical points, the evolution in time of the amplitude A of a finite size disturbance can be described by

$$\frac{\mathrm{d}A}{\mathrm{d}t} = a_0 A + a_1 A^3, \tag{1}$$

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an equation also proposed by Landau [8]. The coefficient a_0 corresponds to the growth rate predicted by the linear theory and a_1 is the first correction (Landau constant) induced by the physical processes arising from nonlinearity. Beside the equilibrium solution of the linear case $(A \to 0)$, this nonlinear equation presents finite amplitude equilibrium solutions (A_e) given by

$$A_e^2 = -\frac{a_0}{a_1}, \quad \text{provided } \frac{a_0}{a_1} < 0. \tag{2}$$

A new unstable equilibrium solution then exists in the subcritical region, $a_0 < 0$, when $a_1 > 0$. While the linear analysis implies that an infinitesimal disturbance will decay, if the amplitude of the disturbance is larger than A_e it will diverge. In other words, at first order the nonlinear effects are destabilising, which is known as subcritical instability, a phenomenon found to occur also in plane Poiseuille flow [9]. For the supercritical region, $a_0 > 0$, a finite amplitude equilibrium solution will exist if $a_1 < 0$, and in that case it will be stable. In consequence infinitesimal disturbances will initially grow exponentially according to the linear theory, but they will eventually saturate to A_e. In the supercritical region then, nonlinear effects can have a stabilising effect, called a supercritical stability, as it is the case of the Taylor vortices that develop in circular Couette flow.

Watson [10] extended Stuart's work and established the scheme called the amplitude expansion method applicable to arbitrary order. Based on the assumption that the disturbance equations are separable and the Landau equation is valid, he proposed a perturbation expansion where the amplitude of the disturbance is the small parameter. In this work we apply this method to determine the value of the Landau constant a_1 along the linear neutral curve. At that point the linear amplification rate a_0 is zero and the stability is determined by a_1. This method has been applied to a variety of flows and the details of the implementation vary. We follow the description of the method by Herbert [11] and Reynolds and Potter [12]. Although there have been proposed several extensions of the procedure to points in the subcritical and supercritical regions, the validity of the expansions away from the linear neutral curve was shown to be very limited [13]. Nevertheless, this type of analysis seems to predict accurately the local bifurcation behaviour, in particular whether the bifurcation is supercritical or subcritical, as was confirmed experimentally in the case of plane Poiseuille flow by Nishioka et al. [14].

2. Governing equations

We consider the Hartmann layer that arises in an electrically conducting flow over an infinite flat surface, perpendicular to a uniform magnetic field B (Fig. 1). The free stream velocity U_0 is in the x direction. The Hartmann layer on the surface can be shown to have thickness $\mathrm{O}(Ha^{-1}L)$ [2], where the Hartmann number is given by

$$Ha = LB\sqrt{\frac{\sigma}{\rho\nu}}, \tag{3}$$

with $\sigma,\ \rho,\ \nu$ the electrical conductivity, density and kinematic viscosity of the fluid, respectively, and L a typical length scale of the flow. We will assume $Ha \gg 1$, which is characteristic of most industrial applications.

Under the usual approximations in magnetohydrodynamics [15], and for small magnetic Reynolds numbers, i.e., assuming that the external magnetic field is unperturbed by the induced magnetic field, the flow is described by

$$\frac{\partial \mathbf{v}^*}{\partial t} + \mathbf{v}^*\nabla\mathbf{v}^* = \frac{1}{\rho}\left(\mathbf{j}^* \times \mathbf{B}^*\right) - \frac{1}{\rho}\nabla p^* + \nu\nabla^2\mathbf{v}^*, \tag{4}$$

$$\nabla \cdot \mathbf{v}^* = 0, \tag{5}$$

$$\mathbf{j}^* = \sigma\left(-\nabla\varphi^* + \mathbf{v}^* \times \mathbf{B}^*\right), \tag{6}$$

$$\nabla \cdot \mathbf{j}^* = 0, \tag{7}$$

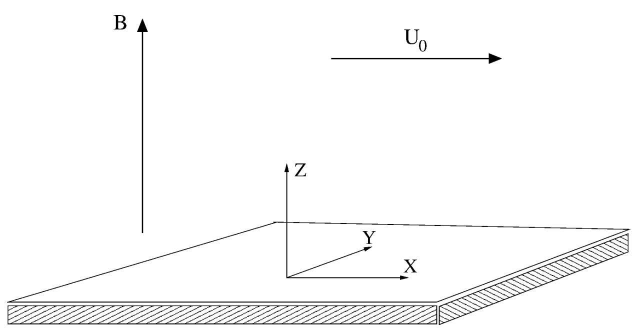

Fig. 1. Model of the flow studied.

where $\mathbf{v}^* = (u^*, v^*, w^*)$ is the velocity vector, $\mathbf{j}^*$ is the current density vector field, p^* the pressure, t the time and φ^* the electric potential field. Following the common approach in stability theory, all the variables are decomposed into a mean steady part (capital letters) and a superimposed disturbance (unstarred variables). In the steady case, the equations containing only basic quantities reduce to (cf. [2])

$$U'' - \frac{\sigma B^2}{\rho\nu} U = -\frac{\sigma B^2}{\rho\nu} U_0, \tag{8}$$

where the primes denote derivatives with respect to z and the boundary conditions correspond to zero velocity at the wall and U_0 far from it. Taking U_0 as the characteristic velocity and the boundary layer thickness (δ) as the characteristic length, the solution of (8) corresponds to an exponential velocity profile,

$$\mathbf{V} = (U, V, W) = \left(1 - \mathrm{e}^{-z}, 0, 0\right). \tag{9}$$

In the case of a solid boundary which is a perfect electrical insulator, the electric currents are restricted or close their path through the Hartmann layers, and U_0 is directly proportional to the total electric current flowing through the layers [2], a characteristic that differentiates them from other boundary layers in classical hydrodynamics and makes their stability properties especially important.

For the perturbations quantities, Eqs. (4), (5) take the form,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{10}$$

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + \frac{\partial U}{\partial z} w + \mathbf{v}\nabla u = -\frac{\partial p}{\partial x} + \nu\nabla^2 u - \frac{\sigma}{\rho} u B^2 - \frac{\sigma}{\rho}\frac{\partial \varphi}{\partial y} B, \tag{11}$$

$$\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} + \mathbf{v}\nabla v = -\frac{\partial p}{\partial y} + \nu\nabla^2 v - \frac{\sigma}{\rho} v B^2 + \frac{\sigma}{\rho}\frac{\partial \varphi}{\partial x} B, \tag{12}$$

$$\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x} + \mathbf{v}\nabla w = -\frac{\partial p}{\partial z} + \nu\nabla^2 w, \tag{13}$$

and Eqs. (6), (7) give

$$\nabla^2\varphi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \tag{14}$$

with the boundary conditions that all the perturbations quantities vanish at the wall and as $z \to \infty$. In the linear approximation, the solution of this set of homogeneous equations can be obtained in the form of a superposition of normal modes, that is plane waves periodic in the x and y directions and of constant frequency,

$$[u, v, w, p, j, \varphi]^{\mathrm{T}} = \left[\hat{u}(z), \hat{v}(z), \hat{w}(z), \hat{p}(z), \hat{j}(z), \hat{\varphi}(z)\right]^{\mathrm{T}} \mathrm{e}^{\mathrm{i}(\alpha x + \beta y - \omega_0 t)}\, \mathrm{e}^{a_0 t}, \tag{15}$$

where α and β are the streamwise and transverse wavenumbers, and ω_0 and a_0 are the frequency and amplification rate that arise as eigenvalues of the linear problem. The sign of a_0 determines whether the amplitude of a mode grows ($a_0 > 0$) or decays ($a_0 < 0$) with time.

By making the change of variables

$$\theta = \alpha x + \beta y, \tag{16}$$

it is possible to adopt a stream function $\psi(\theta, z, t)$ satisfying

$$\frac{\partial \psi}{\partial z} = \alpha u + \beta v, \qquad \frac{\partial \psi}{\partial \theta} = -w, \tag{17}$$

so that Eqs. (10)–(14) become partially decoupled and an equation for the stream function only is obtained, which in non-dimensional form is

$$\frac{\partial \nabla^2\psi}{\partial t} + \alpha U\frac{\partial \nabla^2\psi}{\partial \theta} - \frac{\partial^2 U}{\partial z^2}\frac{\partial \psi}{\partial \theta} + \frac{1}{R}\nabla^2\nabla^2\psi - \frac{1}{R}\frac{\partial^2\psi}{\partial z^2} = \frac{\partial \psi}{\partial \theta}\frac{\partial \nabla^2\psi}{\partial z} - \frac{\partial \psi}{\partial z}\frac{\partial \nabla^2\psi}{\partial \theta}, \tag{18}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial z^2} + k^2\frac{\partial^2}{\partial \theta^2} \quad \text{and} \quad k^2 = \alpha^2 + \beta^2, \tag{19}$$

and R is the Reynolds number based on the boundary layer thickness (δ),

$$R = \frac{Re}{Ha}, \quad \text{with } Re = \frac{U_0 L}{\nu}. \tag{20}$$

The characteristic time, pressure, magnetic field and electric current density scales used are δ/U_0, ρU_0^2, B and $\sigma U_0 B$, respectively. The minimum critical Reynolds number R_c corresponds to the minimum value of R for which one of the eigenmodes of the linear problem is unstable (fundamental) and all the rest are damped, for all the values of α and β.

We take the set of eigenfunctions of the linear equations as a basis for expanding the solution of the nonlinear problem. Near criticality, all the eigenmodes are damped except for the fundamental which becomes unstable. In the weakly nonlinear approach it is assumed that the solution of the nonlinear problem near the bifurcation point can be approximated by a perturbation expansion around this mode. This is in the same spirit of the centre manifold approach [16], which was compared in detail with the amplitude expansion method by Fujimura [17,18].

Due to the nonlinear terms, the fundamental interacts with itself and its complex conjugate, and with the basic flow. At first order, this generates a harmonic, a distortion of the basic flow and of the fundamental. We take as the small parameter in the expansion the magnitude of the fundamental which we now describe as a general function of time $A(t)$, and propose the expansion

$$\psi(\theta, z, t) = \sum_{n=-\infty}^{\infty} A^{|n|}\, \mathrm{e}^{\mathrm{i}n(\theta-\gamma(t))}\psi_n(z, t), \tag{21}$$

where ω_0 is now replaced by a real function $\gamma(t)$ to allow for changes in the frequency with the amplitude. Since our interest centres on the real part of the solution, without loss of generality it is possible to set

$$\psi_{-n}(z, t) = \tilde{\psi}_n(z, t), \tag{22}$$

where the tildes denote complex conjugates. Watson [10] showed that, because the nonlinear terms in (18) are quadratic, as $A \to 0$ the $\psi_n(z, t)$ must be either $\mathrm{O}(1)$, for $n > 0$, or $\mathrm{O}(A^2)$, for $n = 0$, and we can consider an expansion of the form

$$\psi_n(z, t) = \sum_{m=0}^{\infty} A^{2m}\phi_{nm}(z). \tag{23}$$

By the same arguments it is necessary to introduce a Poincaré stretching of the eigenvalues

$$\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}t} = \sum_{\nu=0}^{\infty} a_\nu A^{2\nu}, \qquad \frac{\mathrm{d}\gamma}{\mathrm{d}t} = \sum_{\nu=0}^{\infty} \omega_\nu A^{2\nu}. \tag{24}$$

Substituting expansions (21), (23) and (24) into (18) and equating terms with equal powers of A, a hierarchy of ordinary differential equations for the functions ϕ_{nm} is obtained, which are of the form

$$L_n\phi_{nm} - \sum_{\nu=0}^{m}\bigl[\bigl[n + 2(m-\nu)\bigr]a_\nu - \mathrm{i}n\omega_\nu\bigr]S_n\phi_{nm-\nu} = N_{nm}, \tag{25}$$

where L_n is the linear operator

$$L_n = \frac{1}{R}\left[S_n^2 - \frac{\partial^2}{\partial z^2}\right] - \mathrm{i}n\alpha\bigl[US_n - U''\bigr], \tag{26}$$

with primes denoting derivatives with respect to z and with

$$S_n = \frac{\partial^2}{\partial z^2} - n^2k^2. \tag{27}$$

The nonlinear terms N_{nm} involve functions obtained from previous equations in the hierarchy,

$$N_{nm} = \sum_{\substack{p,q=-\infty \\ r,s=0}}^{\infty} N(\phi_{pr}, \phi_{qs}), \tag{28}$$

with

$$N(\phi_{pr}, \phi_{qs}) = \bigl[k^2pq^2\phi_{pr}\phi'_{qs} + p\phi_{pr}\phi'''_{qs} - k^2q^3\phi'_{pr}\phi_{qs} - q\phi'_{pr}\phi''_{qs}\bigr], \tag{29}$$

and the constraints in the indices

$$p + q = n, \qquad \frac{|p| + |q| - (p+q)}{2} + r + s = m. \tag{30}$$

In the present notation ϕ_{00} corresponds to the stream function of the steady basic state which is function only of the z coordinate. For $n > 0$ the boundary conditions are

$$\phi_{nm}(0) = \phi'_{nm}(0) = \phi_{nm}(\infty) = \phi'_{nm}(\infty) = 0. \tag{31}$$

At O(A), (25) reduces to a Orr–Sommerfeld type equation with solutions ϕ_{10} in the form of normal modes. Since the equations and boundary conditions are homogeneous, the solution is determined apart from an arbitrary multiplicative factor that has to be made definite by imposing a normalisation condition. We chose this to be $\|\phi_{10}\|_{\max} = 1$. This arbitrary normalisation determines the value of a_1 but not its sign, and in consequence it does not affect the character of the bifurcation.

At O(A^2), the interaction of the fundamental with its complex conjugate results in the first correction to the basic flow ϕ_{01}, and the interaction of the fundamental with itself produces the first harmonic ϕ_{20}.

The first correction to the fundamental appears at O(A^3) where the forcing term in (25) now has contributions from the interaction of the fundamental with the first harmonic and with the correction to the basic flow, and from the first nonlinear term in expansions (24). For the points on the linear neutral curve with $a_0 = 0$ the value of a_1 can be determined by making use of a solvability condition. Denoting by $\phi_{10}^{\dagger}$ the eigenfunction of the adjoint operator of L_1,

$$L_1^{\dagger} = \frac{1}{R}\left[S_1^2 - \frac{\partial^2}{\partial z^2}\right] - \mathrm{i}\alpha\left[U S_1 + 2U'\frac{\partial^2}{\partial z^2}\right], \tag{32}$$

and defining the inner product $\langle\,,\rangle$ as

$$\langle f, g\rangle = \int_0^{\infty} \tilde{f}_1(z) g_1(z)\,\mathrm{d}z, \tag{33}$$

it is readily obtained using the properties of the adjoint operator:

$$a_1 - \mathrm{i}\omega_1 = \frac{\langle\phi_{10}^{\dagger}, N_{11}\rangle}{\langle\phi_{10}^{\dagger}, S_1\phi_{10}\rangle}. \tag{34}$$

3. Numerical results and discussion

Given the extensive numerical calculations involved in obtaining the values of a_1, and to prevent a dramatic error propagation, an accurate numerical method had to be used. The hierarchy of ordinary differential equations described in Section 2 were discretised using the Tau spectral method [19] and expansions in Chebyshev polynomials. The resulting linear algebra problems were solved using routines from the LAPACK numerical library [20]. The transformation of the semi-infinite domain into the domain of definition of the Chebyshev polynomials $[-1, 1]$ was achieved by means of an algebraic mapping, and the integrals arising from the inner products in the equation for the Landau constant (34) were approximated numerically with the Clenshaw and Curtis method [21].

We restricted the analysis to two dimensional perturbations because Squire's theorem [22] applies to this problem and the minimum critical Reynolds number corresponds to the case $\beta = 0$. The neutral curve for this mode is shown in Fig. 2. Our calculations yielded a critical Reynolds number of 48257 for $\alpha = 0.16$ and $\omega_0 = 0.025$. Lingwood and Alboussière [4] studied the linear stability of the Hartmann layer when the magnetic field is not strictly perpendicular to the solid boundary. They found that for a constant magnitude of the field in the $\hat{z}$ direction, a nonzero component in the $\hat{x}$ direction (the direction of the flow) has a very small stabilising effect, while no differences in the stability were found when the $\hat{y}$ component of B is changed. Given that the weakly nonlinear analysis is built around the most unstable mode from the linear analysis, and because this appears to have little dependence on the $\hat{x}$ and $\hat{y}$ components of B, here we restricted the analysis to the case when the magnetic field is exactly perpendicular to the solid boundary.

At the critical point the weakly nonlinear analysis gives $a_1 = 52.08$ and $\omega_1 = -28.17$, indicating subcritical instability. In Figs. 3–6 are shown the functions involved in obtaining these values. The distribution in sign of a_1 along the neutral curve is shown in Fig. 2, where it can be seen that the bifurcation remains subcritical along the upper branch and a part of the lower branch close to the critical point. This different behaviour in the upper and lower branches was also found in the paradigmatic cases of plane Poiseuille and Blasius flow [23,24]. The variations in magnitude of a_1 and ω_1 along the neutral curve are shown in Fig. 7. According to (2) the larger the value of a_1 the smaller the magnitude of A_e necessary for a disturbance to grow in the subcritical region. This indicates that perturbations of small amplitude can trigger nonlinear effects at an early stage and consequently laminarisation and transition should occur in the subcritical region. In the proximity of the minimum critical Reynolds number it was found that $a_0 \approx 1.7 \times 10^{-8}(48257 - R)$. Assuming that the value of a_1 varies slowly in the proximity

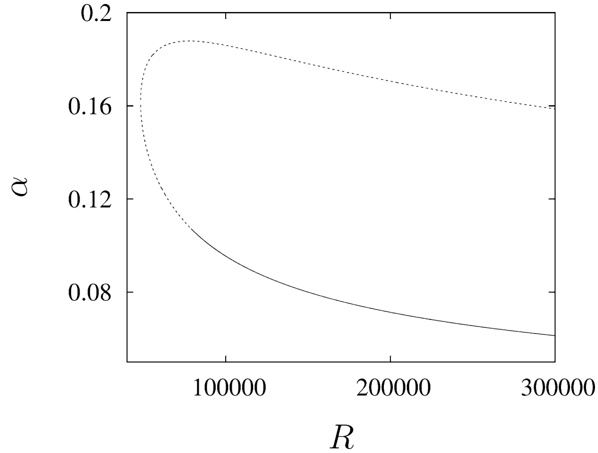

Fig. 2. Sign distribution of a_1 along the linear stability neutral curve. The continuous and dashed lines correspond to supercritical and subcritical instability respectively.

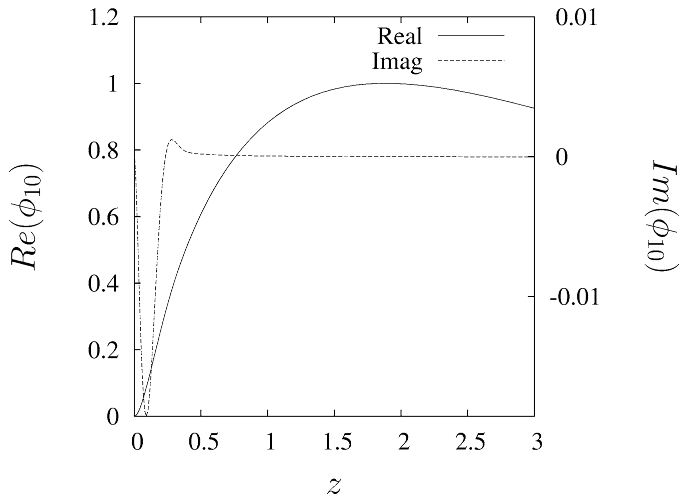

Fig. 3. Real and imaginary parts of the eigenfunction of the linear problem (fundamental).

of R_c and taking into account the normalisation criterion previously introduced, an estimate is obtained for a lower bound of the intensity of the perturbation (at the point where $\|\phi_{10}\|_{\max}$) corresponding to the finite amplitude unstable equilibrium solution in the subcritical region:

$$\left(\overline{w^2}\right)^{1/2} \approx 2 \times 10^{-5}(48257 - R)^{1/2}, \tag{35}$$

which shows that very weak disturbances relative to U_0 can trigger nonlinear effects in the subcritical region.

By means of an energetic analysis, which involves the fully nonlinear equations, Lingwood and Alboussière [4] showed that the transfer of energy from the basic flow into the disturbance is possible for $R > 26$. According to Reddy et al. [25] this implies that infinitesimal disturbances can be amplified for a finite period of time in the linearly stable region, before being eventually damped. This transient growth, although not exponential but algebraic, has been verified in numerical simulations by Gerard-Varet [26], who showed that the amplification factor can be as high as 500 for $R = 1000$ and oblique waves. The possibility of such an important growth shows that disturbances of even smaller size than A_e can eventually reach such an amplitude as to activate nonlinear effects and instability. This is compatible with the experimental results that found laminarisation of magnetically driven flows at $R \ll R_c$, giving further support to the thesis that stability in these flows is determined by the Hartmann layers and nonlinear effects. It is also to be expected that transition from the laminar to a turbulent state will be initiated at subcritical values of R, although no experimental results are available at present for comparison.

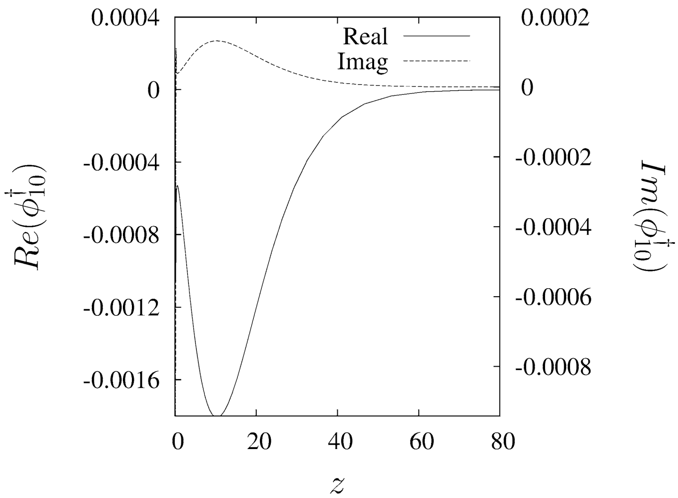

Fig. 4. Real and imaginary parts of the eigenfunction of the adjoint linear operator.

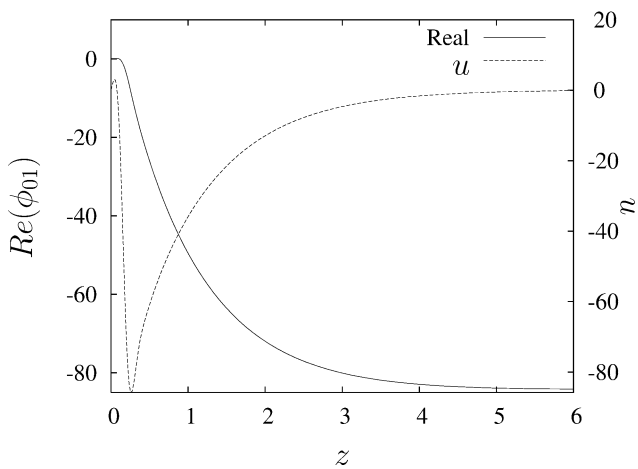

Fig. 5. First correction to the basic flow ($u = \phi'_{01}$).

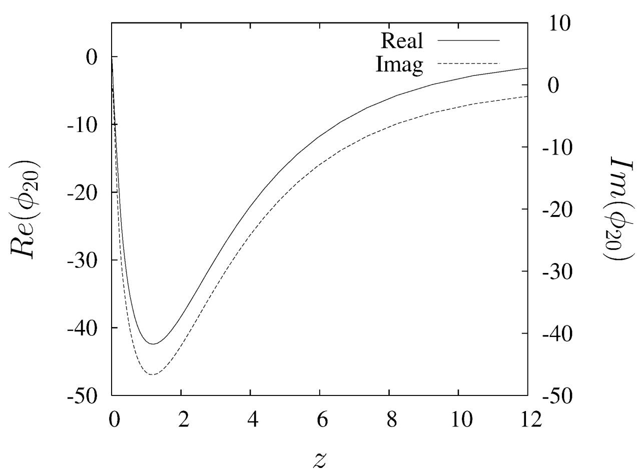

Fig. 6. Real and imaginary parts of the first harmonic.

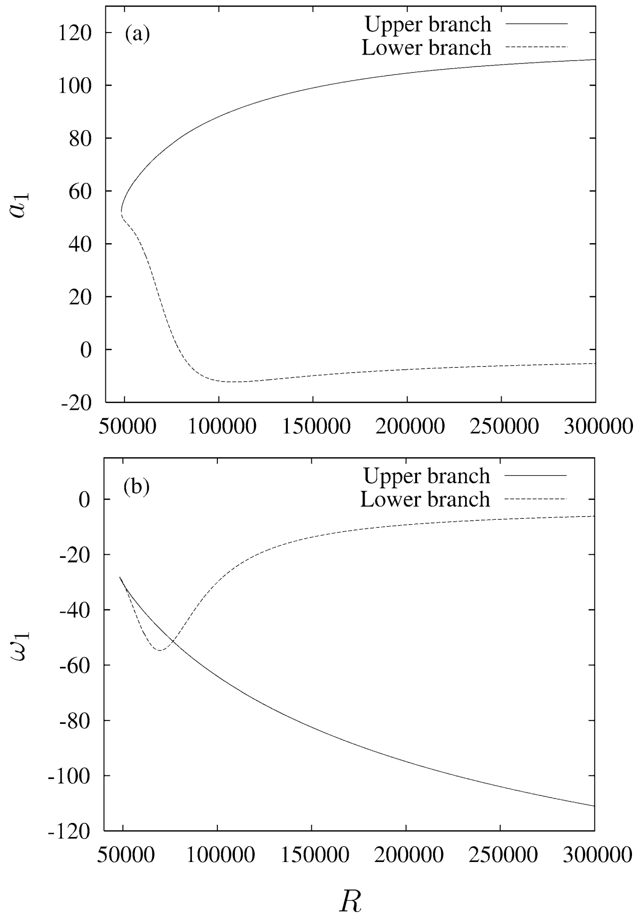

Fig. 7. Distribution of a_1 (a) and ω_1 (b) along the neutral curve.

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